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Short communication

Estimating the onset of natural convection in a horizontal layer of a fluid with a temperature-dependent viscosity

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Abstract

A simple method, which can be applied to estimate the onset of natural convection in a fluid with a temperature-dependent viscosity, is presented here. It is shown that this method is very useful for experimentalists and engineers to estimate the onset of natural convection in a horizontal fluid layer.

Keywords: Natural convection; Temperature-dependent viscosity; Linear instability theory

1. Introduction

Natural convection in a fluid layer with a strongly temperature-dependent viscosity has received considerable attention in recent years, because of its wide applications in research and industry fields such as geophysics, astrophysics, material science, solar collectors, heat exchangers, nuclear reactors and so on. Numerous studies have shown that strongly temperature-dependent viscosity has a great effect on the onset of natural convection. Some publications about this topic are Stengel et al. [1], Busse and Frick [2], White [3] and Bottaro et al. [4].

The effect of temperature-dependent viscosity on the onset of natural convection is reconsidered in this communication. The propose of this study is to look for a simple way to estimate the onset of natural convection for a system, where the viscosity of working fluid is strongly temperature-dependent. The method described here is very useful for experimentalists and engineers, especially, when they design new convection systems.

2. Mathematical description of the problem

We consider a horizontal infinite fluid layer of thickness H^* with the temperature T_1^* and T_2^* at the upper and lower boundaries, respectively. Except the dependence of viscosity μ^* on temperature and the linear dependence of density on temperature in gravity term (Boussinesq approximation), all other properties are regarded as constants. The density ρ^* (in gravity term only) is expressed as

$$\rho^* = \rho_R^*[1 - \alpha^*(T^* - T_R^*)], \tag{1}$$

where α^* is the thermal expansion coefficient, R the reference point which is located on the midplane of the fluid layer and the reference temperature is $T_R^* = 1/2(T_1^* + T_2^*)$. All dimensional quantity is stared.

The present analysis is based on an extended version of linear stability theory which holds when viscosity is a function of temperature. In this theory, quantity a^* is decomposed into a mean value $\bar{a}^*$ and a disturbance $\tilde{a}^*$ as

$$a^* = \bar{a}^* + \tilde{a}^*, \tag{2}$$

here a^* represents velocity, pressure, temperature and viscosity.

We look for a steady cellular solution as Stengel did [1]. The disturbance $\tilde{a}^*$ is assumed as

$$\tilde{a}^* = \hat{a}^*(z^*)\,e^{(ik_x^*x^*+ik_y^*y^*)} + \text{c.c.}, \tag{3}$$

where c.c. stands for the complex conjugate.

Using $H^*, H^{*2}/\kappa^*$ and $\Delta T^* = T_2^* - T_1^*$ as the scales for the length, time and temperature, respectively and substituting

$$p^* = \rho_R^* g^* H^* \bar{\Pi}_1(z) + \frac{\mu_R^*\kappa^*}{H^{*2}}\tilde{\Pi}_2(x, y, z, t) \quad \text{and}$$

$$T^* = T_R^* + \Delta T^*[\bar{T}(z) + \tilde{T}(x, y, z, t)]$$

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Nomenclature

a	general quantity
C, d	viscosity law constants
H	channel height
k	wave number
k_x	x-component wave number
k_y	y-component wave number
p	pressure
Ra	Rayleigh number
Rc	system parameter
s	the largest relative error
T	temperature
w	velocity normal to the plate
x, y, z	Cartesian coordinates

Greek symbols

α	thermal expansion coefficient
ε	heat transfer rate
κ	thermal conductivity
μ	viscosity
ρ	density

Subscripts

c	critical value
R	reference state

Superscripts

*	dimensional quantity
$\wedge$	shape function

together with Eqs. (1)–(3) into the N–S equations which hold for variable viscosity and the energy equation, after canceling the x and y component of velocity and pressure disturbances, the non-dimensionalised disturbance equations are

$$\left\{\left[\bar{\mu}\left(\frac{d^2}{dz^2}-k^2\right)+2\frac{d\bar{\mu}}{dz}\frac{d}{dz}\right]\left(\frac{d^2}{dz^2}-k^2\right) + \frac{d^2\bar{\mu}}{dz^2}\left(\frac{d^2}{dz^2}+k^2\right)\right\}\hat{w} - Rak^2\hat{T} = 0, \quad (4)$$

$$\left(\frac{d^2}{dz^2}-k^2\right)\hat{T} - \frac{d\bar{T}}{dz}\hat{w} = 0, \quad (5)$$

where $k^2 = k_x^2 + k_y^2$. The z-axis of the Cartesian co-ordinate system coincides with the opposite direction of gravity. Rayleigh number (Ra) is $Ra = g^*\rho_R^*\alpha^*\Delta T^* H^{*3}/\mu_R^*\kappa^*$. The mean temperature field is $\bar{T}(z) = -z$.

The boundary conditions at $z = \pm 1/2$ with fixed plate temperature are

$$\hat{w} = \frac{d\hat{w}}{dz} = \hat{T} = 0. \quad (6)$$

3. The introduction of new parameter

It is well known that natural convection in a horizontal fluid layer with isothermal boundaries and no internal heat sources is governed by Ra. The definition of Ra gives the following information:

- Ra depends on the temperature difference between the two plates ΔT^*.
- Ra depends on the gravity acceleration, the characteristic length H^*, the reference temperature T_R^* and the properties of working fluid at T_R^*.

Now we recall Ra and separate it into two parts as

$$Ra = \frac{g^*\rho_R^*\alpha^*\Delta T^* H^{*3}}{\mu_R^*\kappa^*} = \varepsilon \mathrm{Rc}, \quad (7)$$

where $\mathrm{Rc} = g^*\rho_R^*\alpha^* T_R^* H^{*3}/\mu_R^*\kappa^*$ is the introduced non-dimensional parameter. It is a intrinsic number for a system and depends on the gravity acceleration, the reference temperature, the characteristic length and the properties of working fluid at the reference temperature. $\varepsilon = \Delta T^*/T_R^*$ is the heat transfer rate. By introducing Rc, Ra is successfully separated the effect of external supplied temperature difference ΔT^* from that of Rc. For starting natural convection, Ra should at least be equal to the critical Rayleigh number (Ra_c). Thus, from Eq. (7), it is found that for a system with large Rc, a small ΔT^* is needed to start natural convection.

4. Results and discussion

The numerical method used to solve Eqs. (4)–(6) is the Chebyshev collocation method. $\hat{T}$ and $\hat{w}$ are expanded in terms of Chebyshev polynomals. In our case, 40 Chebyshev polynomals are appropriate. Fig. 1 shows our results agree very well with those of Busse and Frick [2], where viscosity depends linearly on temperature and r is the ratio of viscosity at the upper boundary to that at the lower boundary.

For common liquids, the temperature-dependent viscosity can be expressed as

$$\mu = C\, e^{d^*/T^*}, \quad (8)$$

where C and d^* are constants, which are determined by the kind of liquid and the reference temperature, respectively.

We firstly use water as an example working fluid in order to show how our method is applied to determine the Ra_c with temperature-dependent viscosity effect. Water is chosen here because its viscosity strongly depends on temperature and its other properties do not. Thus, all properties are considered as constants except the viscosity and the density in the gravity term. For water, $C = 0.002243$ and $d^* = 1787.33$ K.

Fig. 2 shows the dependence of Ra_c on ε for water (curve AB). Then the parameter Rc should be determined for the convection system. For example, if $\mathrm{Rc} = 20\,000$ for this system, we can draw the straight line $Ra_c = \mathrm{Rc}\varepsilon$ (line OC)

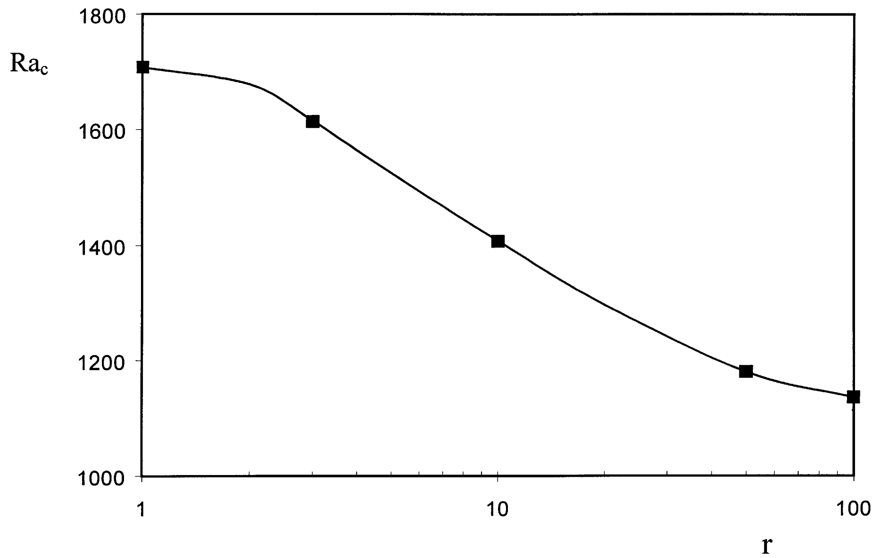

Fig. 1. The critical Rayleigh number as a function of the viscosity ratio *r*. The solid square points are our numerical results, the curve is the result of Busses and Frick [2].

in Fig. 2. The cross point of curve AB and line OC is D. The abscissa and vertical coordinates of the point D show the critical heat transfer rate ε_c and Ra_c for this system.

There is a important point that we should notice. For all liquids, they have freezing point and boiling point. If the upper or the below plate temperature is out of the liquid state temperature range of the working liquid, the viscosity law (Eq. (8)) of liquid does not hold and our method cannot be applied. For example, the water's freezing point is 273 K. Then the largest ΔT^* is 40 K for the system with a reference temperature 293 K. Thus, the largest possible ε is 0.1365 (point E). If the cross point D is on the right side point E, it means $\Delta T^* > 40$ K and the water near the upper plate is freezing.

The method to estimate Ra_c as shown in Fig. 2 gives a relatively low resolution result. To improve its resolution, Eq. (7) is rewritten as

$$Ra_c - Ra_{c0} = \mathrm{Rc}\left(\varepsilon - \frac{Ra_{c0}}{\mathrm{Rc}}\right) \quad (9)$$

where $Ra_{c0} = 1707.8$ is the critical Rayleigh number with constant viscosity.

From the known relation of critical Rayleigh number and heat transfer rate, i.e. $Ra_c = f(\varepsilon)$, the curve which shows

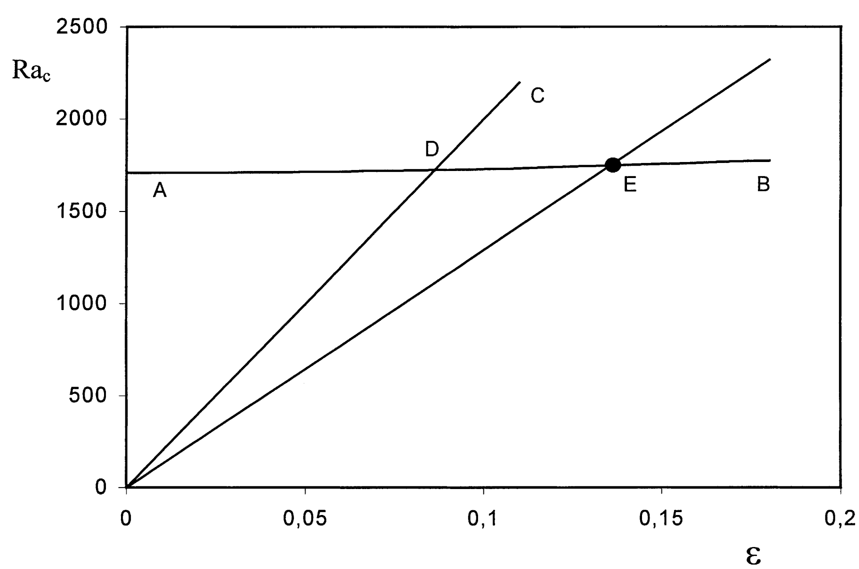

Fig. 2. The dependence of the critical Rayleigh number on heat transfer rate at $T_R^* = 293$ K for water. The procedure to estimate the critical Rayleigh number for Rc = 20000 system.

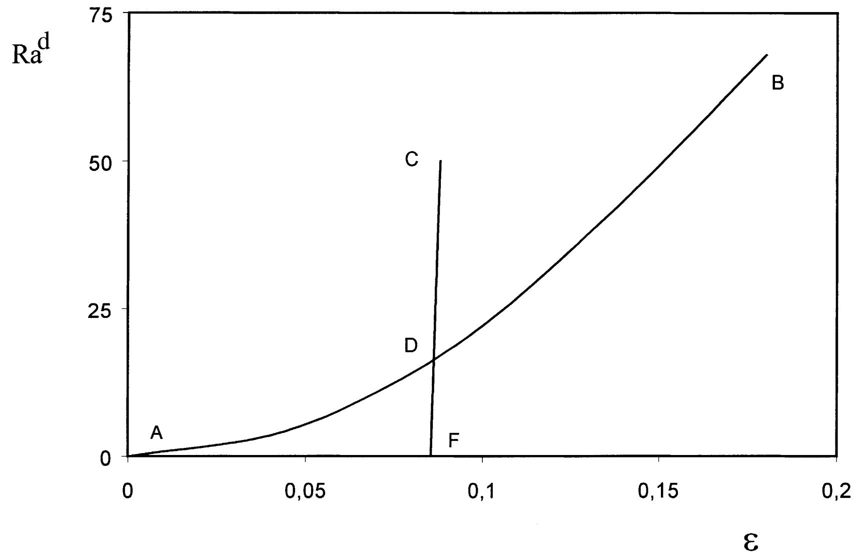

Fig. 3. The difference of the critical Rayleigh number and the critical Rayleigh number with constant viscosity depends on heat transfer rate at $T_R^* = 293$ K for water. The procedure to estimate the critical Rayleigh number for Rc = 20000 system by the method with high resolution.

the dependence of $Ra^d = Ra_c - Ra_{c0}$ on ε for water (curve AB) and the straight line $Ra^d = \mathrm{Rc}(\varepsilon - Ra_{c0}/\mathrm{Rc})$ for Rc = 20000 (line FC) in Fig. 3 can be drawn. The cross point D of curve AB and line FC is (Ra_D^d, ε_D). The critical Rayleigh number and the critical heat transfer rate for this system are $Ra_c = Ra_{c0} + Ra_D^d$ and $\varepsilon_c = \varepsilon_d$, respectively.

The basic steps of applying our method are:

1. Determine the reference temperature $T_R^* = 1/2(T_1^* + T_2^*)$.
2. Find the relation of critical Rayleigh number Ra_c and heat transfer rate ε at T_R^*.
3. Draw the dependence of $Ra^d = Ra_c - Ra_{c0}$ on ε (like the curve AB in Fig. 3).
4. Calculate the Rc number of the convection system.
5. Draw the line $Ra^d = \mathrm{Rc}(\varepsilon - Ra_{c0}/\mathrm{Rc})$ in the same co-ordinate system of step (3).
6. Look for the cross point D coordinates (Ra_D^d, ε_D) and compute the Ra_c and critical heat transfer rate by $Ra_c = Ra_{c0} + Ra_D^d$ and $\varepsilon_c = \varepsilon_d$, respectively.
7. Finally check if the viscosity law (Eq. (8)) can be applied for this case.

From the above steps, it is found that the core procedure is the step (2). If the relation $Ra_c = f(\varepsilon)$ at the reference temperature is found, Ra_c and critical heat transfer rate can be known easily. Now we will explain how to find the relation $Ra_c = f(\varepsilon)$ from a known database.

Table 1 gives the Ra_c s for different liquids and heat transfer rates at two reference temperatures. For the liquid which is not shown in Table 1, it is found that the relation of Ra_c and heat transfer rate can be determined by linear interpolation from the results of its two neighbour d^*s. For example, if we want to look for the relation of Ra_c and heat transfer rate at $d^* = 4700$ K, we apply the linear interpolation and have

$$Ra_c(4700, \varepsilon) = \frac{Ra_c(5000, \varepsilon)(d^* - 4000) + Ra_c(4000, \varepsilon)(5000 - d^*)}{1000}. \quad (10)$$

Then the curve of Ra_c with respect to ε can be found for $d^* = 4700$ K if we know Ra_c values at several heat transfer rates. Table 2 shows the results by the linear interpolation

Table 1
The critical Rayleigh number

	0 K	1000 K	2000 K	3000 K	4000 K	5000 K
$T_R^* = 293$ K						
0.1	1707.8	1718.1	1733.6	1754.4	1780.0	1810.1
0.2	1707.8	1749.2	1811.5	1891.6	1985.1	2085.7
0.3	1707.8	1802.2	1941.2	2109.0	2281.7	2429.1
0.4	1707.8	1878.5	2121.1	2382.1	2586.7	2667.3
$T_R^* = 323$ K						
0.1	1707.8	1716.9	1730.4	1748.1	1770.0	1795.6
0.2	1707.8	1744.5	1798.4	1867.7	1949.3	2038.5
0.3	1707.8	1791.4	1912.6	2060.7	2218.8	2365.6
0.4	1707.8	1859.0	2072.7	2311.8	2522.1	2648.4

Table 2
The critical Rayleigh number by linear interpolation and exact solution method, $d^* = 4700$ K

	$T_R^* = 293$ K		$T_R^* = 323$ K	
ε	Interpolation	Exact	Interpolation	Exact
0.1	1801.1	1800.6	1787.9	1787.5
0.2	2055.5	2055.4	2011.8	2011.2
0.3	2384.5	2389.2	2321.5	2324.1
0.4	2643.1	2658.9	2610.5	2622.1

method and the results by solving Eqs. (4)–(6) with viscosity law (8) for $d^* = 4700\,\mathrm{K}$. The method to solve Eqs. (4)–(6) with viscosity law (8) is called "exact solution method" here.

The largest relative errors of Ra_c between the linear interpolation results and the exact solution results is less than 0.65% in Table 2. We further find that s is less 1% for all other d^* ($d^* < 5000\,\mathrm{K}$ and $\varepsilon < 0.4$). Thus, it is concluded that the dependence of Ra_c on heat transfer rate for all $d^*(0 < d^* < 5000)$ at the reference temperature (293 or 323 K) can be determined by the linear interpolation method from the database shown in Table 1.

By the same way, we also find that the dependence of Ra_c on heat transfer rate at a reference temperature T_R^* ($293 < T_R^* < 323\,\mathrm{K}$) can be determined by linearly interpolating the corresponding results (shown in Table 1) at reference temperatures 293 and 323 K. The results is not presented here for brief.

In the above discussions, it is showed how to determine the relation of Ra_c and heat transfer rate for a wide range of d^* and T_R^* ($0\,\mathrm{K} < d^* < 5000\,\mathrm{K}$ and $293\,\mathrm{K} < T_R^* < 323\,\mathrm{K}$) by the linear interpolation method. Thus, Ra_c can be determined by our method for all systems with working liquid and reference temperature in the above range. It is further concluded that if we have a database with the information for all liquids and reference temperatures, the critical Rayleigh number can be determined by our method for all systems with temperature-dependent viscosity effect.

5. Conclusions

The effect of temperature-dependent viscosity on Ra_c is further clarified by introducing the parameter Rc. The present method shows a rather simple way to estimate the critical Rayleigh number with the influence of temperature-dependent viscosity. There is no doubt that this method is welcomed by the experimentalists and engineers who want to know the Ra_c for a convection system. This method can be further generated to consider the effects of other temperature-dependent properties such as density, heat conductivity and specific heat. On the other hand, the present method can also be applied to estimate the onset of natural convection with other boundary conditions.

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